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——一阶微分形式不变性

7.3 多元复合函数求导法

一元复合函数 $y = f(u),\ u = \varphi(x)$

$u = \varphi(x)$在点x处可导，而$y = f(u)$在与x相对应的点u处可导，则

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

链式法则

变量关系图：

y —— u —— x

多元复合函数的几类情形:

(1) $\begin{cases} z = f(u,v) \\ u = \varphi(t) \\ v = \psi(t) \end{cases}$ $z = f[\varphi(t),\psi(t)]$ 求 $\dfrac{dz}{dt}$

（2） $z = f(u),\ u = \varphi(x, y)$ 求 $\dfrac{\partial z}{\partial x}$ 和 $\dfrac{\partial z}{\partial y}$

(3) $\begin{cases} z = f(u,v) \\ u = \varphi(x, y) \\ v = \psi(x, y) \end{cases}$ $z = f[\varphi(x, y),\psi(x, y)]$

特殊: $\begin{cases} z = f(u, x, y) \\ u = \varphi(x, y) \end{cases}$ $z = f[\varphi(x, y), x, y]$

7.3.1 多元与一元的复合

1 自变量只有一个的情况

定理 7.3.1 如果函数 $u=\phi(t)$ 及 $v=\psi(t)$ 都在点 t 可导，函数 $z=f(u,v)$ 在对应点 (u,v) 具有连续偏导数，则复合函数 $z=f[\phi(t),\psi(t)]$ 在对应点 t 可导，且其导数可用下列公式计算：

$$\frac{dz}{dt}=\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}.$$

证明 设 t 获得增量 Δt,

则 $\Delta u=\phi(t+\Delta t)-\phi(t)$, $\Delta v=\psi(t+\Delta t)-\psi(t)$;

由于函数$z=f(u,v)$在点(u,v)有连续偏导数

$$\Delta z=\frac{\partial z}{\partial u}\Delta u+\frac{\partial z}{\partial v}\Delta v+\varepsilon_1\Delta u+\varepsilon_2\Delta v,$$

当$\Delta u\to 0$，$\Delta v\to 0$时，　$\varepsilon_1\to 0$，$\varepsilon_2\to 0$

$$\frac{\Delta z}{\Delta t}=\frac{\partial z}{\partial u}\cdot\frac{\Delta u}{\Delta t}+\frac{\partial z}{\partial v}\cdot\frac{\Delta v}{\Delta t}+\varepsilon_1\frac{\Delta u}{\Delta t}+\varepsilon_2\frac{\Delta v}{\Delta t}$$

$u=\varphi(t), v$可导$\Rightarrow$当$\Delta t\to 0$时，$\Delta u\to 0$，$\Delta v\to 0$

$$\lim_{\Delta t\to 0}\frac{\Delta u}{\Delta t}=\frac{du}{dt},\qquad \lim_{\Delta t\to 0}\frac{\Delta v}{\Delta t}=\frac{dv}{dt},$$

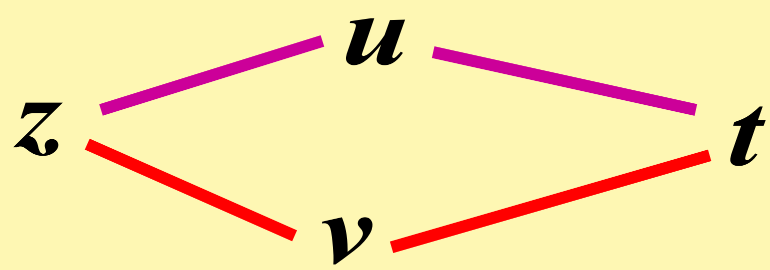
$$\lim_{\Delta t\to 0}\left(\varepsilon_1\frac{\Delta u}{\Delta t}+\varepsilon_2\frac{\Delta v}{\Delta t}\right)=0$$

$$\begin{cases} z = f(u,v) \\ u = \varphi(t) \\ v = \psi(t) \end{cases} \qquad z = f[\varphi(t),\psi(t)]$$

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t}$$

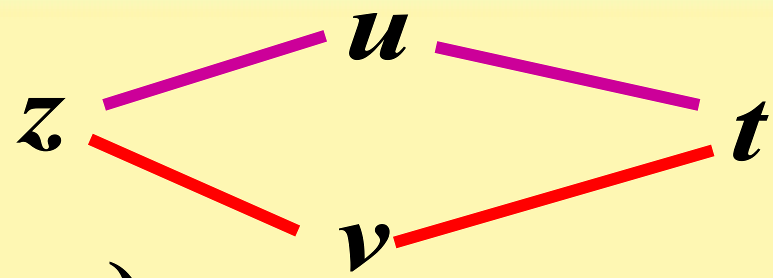
$$= \lim_{\Delta t \to 0}\left(\frac{\partial z}{\partial u}\cdot\frac{\Delta u}{\Delta t} + \frac{\partial z}{\partial v}\cdot\frac{\Delta v}{\Delta t}\right) + \lim_{\Delta t \to 0}\left(\varepsilon_1\frac{\Delta u}{\Delta t} + \varepsilon_2\frac{\Delta v}{\Delta t}\right)$$

$$= \frac{\partial z}{\partial u}\cdot\frac{du}{dt} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dt}$$

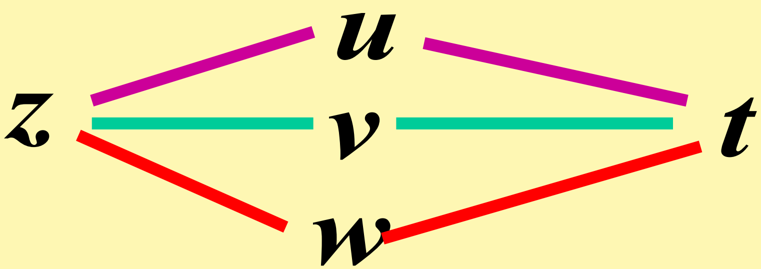


公式中的导数 $\frac{dz}{dt}$ 称为**全导数.**

$$\frac{dz}{dt}=\frac{\partial z}{\partial u}\cdot\frac{du}{dt}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dt}$$



推广：例如 $z=f(u,v,w)$



链式法则：连线相乘，分线相加

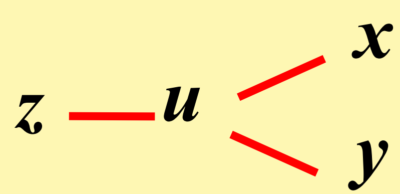
$$\frac{dz}{dt}=\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}+\frac{\partial z}{\partial w}\frac{dw}{dt}=f_u'u_t+f_v'v_t+f_w'w_t$$

$$=f_1'u_t+f_2'v_t+f_3'w_t=f_uu_t+f_vv_t+f_ww_t$$

$$=f_1u_t+f_2v_t+f_3w_t$$

2 当仅有一个中间变量时

$z = f(u), u = \varphi(x, y)$　复合函数 $z = f[\varphi(x, y)]$

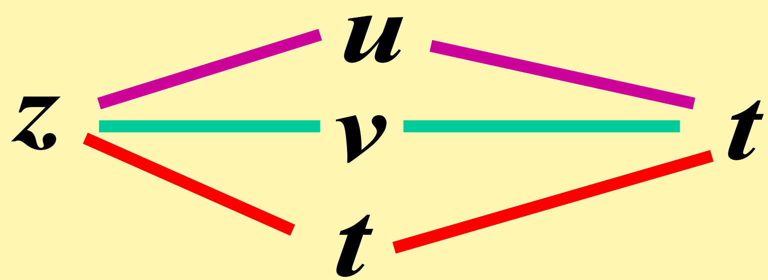


$$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{df}{du} \cdot \frac{\partial \varphi}{\partial x} = f'(u) \cdot \varphi_x,$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = \frac{df}{du} \cdot \frac{\partial \varphi}{\partial y} = f'(u) \cdot \varphi_y$$

例1　设$z = uv + \sin t$，而$u = e^t$，$v = \cos t$，

求全导数$\dfrac{dz}{dt}$.



解法一： $z = e^t \cos t + \sin t$ 转化为一元函数的求导

解　$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\cdot\frac{du}{dt} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dt} + \frac{\partial z}{\partial t}\cdot\frac{dt}{dt}$$

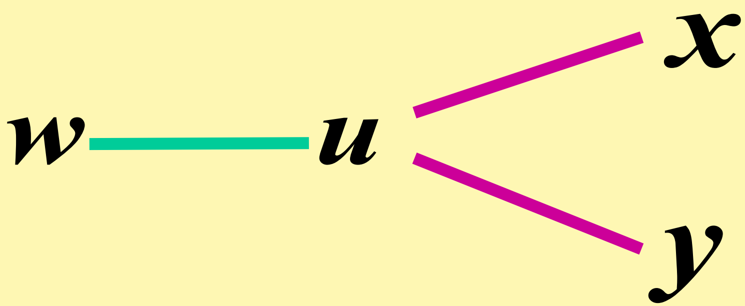
$$= ve^t - u\sin t + \cos t = e^t \cos t - e^t \sin t + \cos t$$

注意：$\dfrac{dz}{dt}$是$z = e^t \cos t + \sin t$对t求导

$\dfrac{\partial z}{\partial t}$是$z = uv + \sin t$对$t$求偏导，视$u,v$为常数

例2 设$z = \dfrac{y}{f(x^2 - y^2)}$,其中$f(u)$可导，验证

$$\frac{1}{x}\frac{\partial z}{\partial x} + \frac{1}{y}\frac{\partial z}{\partial y} = \frac{z}{y^2}$$



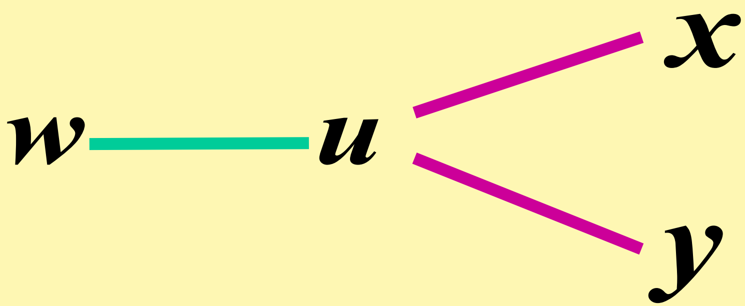
分析 原函数由$z = \dfrac{y}{w}$, $w = f(u)$, $u = x^2 - y^2$复合

证明 记$f' = \dfrac{df}{du}$　　$\dfrac{\partial w}{\partial x} = \dfrac{dw}{du} \cdot \dfrac{\partial u}{\partial x} = f' \cdot 2x$

$$\frac{\partial z}{\partial x} = y \cdot \frac{-1}{w^2} \cdot \frac{\partial w}{\partial x} = -\frac{yf' \cdot 2x}{f^2}$$

例2 设$z=\dfrac{y}{f(x^2-y^2)}$,其中$f(u)$可导，验证

$$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{z}{y^2}$$



分析 原函数由$z=\dfrac{y}{w}$，$w=f(u)$, $u=x^2-y^2$复合

$$\frac{\partial w}{\partial y}=\frac{dw}{du}\cdot\frac{\partial u}{\partial y}=-f'\cdot 2y$$

$$\frac{\partial z}{\partial y}=\frac{1}{w}-y\cdot\frac{1}{w^2}\cdot\frac{\partial w}{\partial y}=\frac{f-y\cdot f'\cdot(-2y)}{f^2}$$

例2 设$z=\dfrac{y}{f(x^2-y^2)}$,其中$f(u)$可导，验证

$$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{z}{y^2}$$

已求得 $\dfrac{\partial z}{\partial x}=-\dfrac{yf'\cdot 2x}{f^2}$ $\qquad \dfrac{\partial z}{\partial y}=\dfrac{f-yf'\cdot(-2y)}{f^2}$

$$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{1}{x}\left(-\frac{yf'\cdot 2x}{f^2}\right)+\frac{1}{y}\left(\frac{f-yf'\cdot(-2y)}{f^2}\right)$$

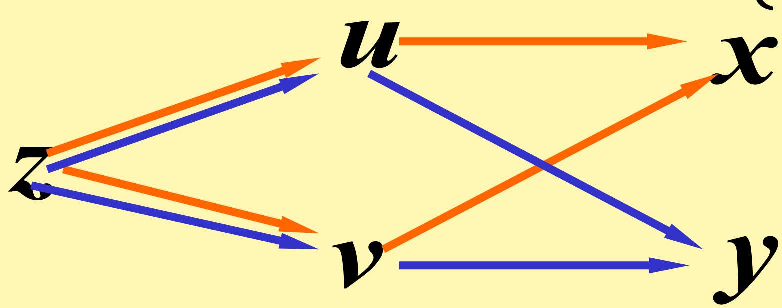
$$=\frac{z}{y^2}$$

7.3.2 多元与多元的复合

定理7.3.1可推广到中间变量是多元函数的情况

$$z = f(u,v), \quad \begin{cases} u = \varphi(x,y) \\ v = \psi(x,y) \end{cases}$$

变量关系图：



链式法则：“连线相乘，分线相加”

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = f_u \cdot u_x + f_v \cdot v_x$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = f_u \cdot u_y + f_v \cdot v_y$$

定理 7.3.2 如果$u=\phi(x,y)$及$v=\psi(x,y)$都在点(x,y)具有对x和y的偏导数，且函数$z=f(u,v)$在对应点(u,v)具有连续偏导数，则复合函数

$z=f[\phi(x,y),\psi(x,y)]$在对应点(x,y)的两个偏导数存在，且可用下列公式计算

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x},\qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

例 3 设$z = e^u \sin v$，而$u = xy$，$v = x + y$，

求 $\frac{\partial z}{\partial x}$和$\frac{\partial z}{\partial y}$.

解 $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = e^u \sin v \cdot y + e^u \cos v \cdot 1$$

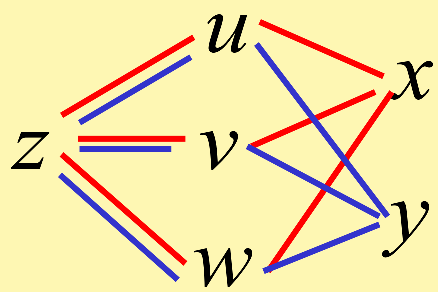
$$= e^u (y\sin v + \cos v) = e^{xy}(y\sin(x+y) + \cos(x+y))$$

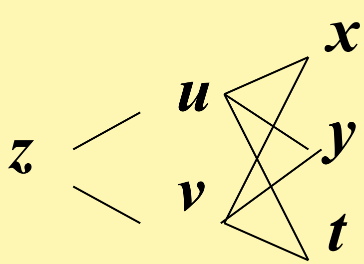
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = e^u \sin v \cdot x + e^u \cos v \cdot 1$$

$$= e^u (x\sin v + \cos v) = e^{xy}(x\sin(x+y) + \cos(x+y)).$$

注 也可由$z=e^{xy}\sin(x+y)$ 而直接对x、y求偏导

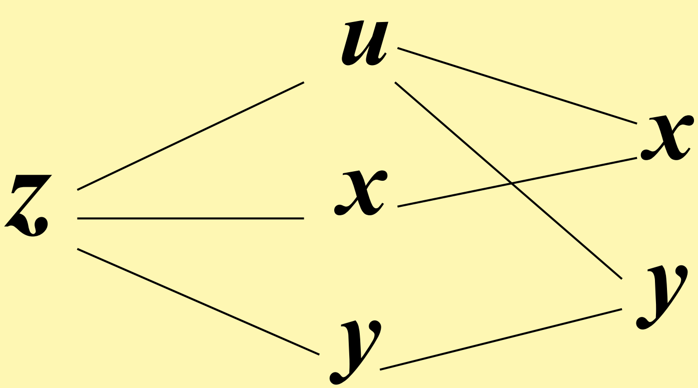
注 1 上述公式可推广：中间变量及自变量的个数可增加或减少.





2 复合函数中自变量与中间变量共存

设 $z=f(u, x, y)$ 具有连续偏导数，$u=\varphi(x,y)$ 具有导数，则 $z=f[\varphi(x,y),x,y]$ 对 x,y 的偏导数为：

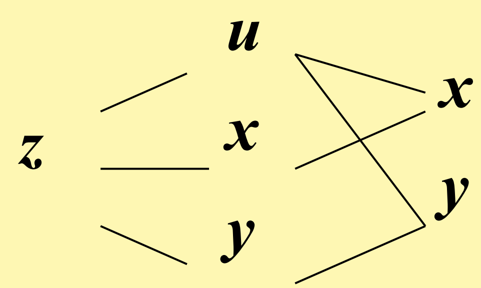


$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial x}\cdot\frac{dx}{dx}$$ ×

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial x}$$

$z=f(u, x, y)$ $u=\varphi(x,y)$, 既有中间变量，又有自变量

把 $\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial x}\cdot\dfrac{dx}{dx}$



写成 $\dfrac{\partial z}{\partial x}=\dfrac{\partial f}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial f}{\partial x}=f_u\cdot u_x+f_x=f_1\cdot u_x+f_2$

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial x},\qquad \frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial y}.$$

两者的区别

把复合函数 $z=f[\varphi(x,y),x,y]$ 中的 y 看作不变而对 x 的偏导数

把 $z=f(u,x,y)$ 中的 u 及 y 看作不变而对 x 的偏导数

例4 $u=f(x,y,z)=e^{x^2+y^2+z^2}$

而$z=x^2\sin y$. 求 $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$

解: $$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}=f_1\cdot 1+f_2\cdot 0+f_3\cdot z_x$$

$$=2xe^{x^2+y^2+z^2}+2ze^{x^2+y^2+z^2}\cdot 2x\sin y$$

$$=2x(1+2x^2\sin^2 y)e^{x^2+y^2+x^4\sin^2 y}$$

$$\frac{\partial u}{\partial y}=\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y}=f_1\cdot 0+f_2\cdot 1+f_3\cdot z_y$$

$$=2ye^{x^2+y^2+z^2}+2ze^{x^2+y^2+z^2}\cdot x^2\cos y$$

$$=2(y+x^4\sin y\cos y)e^{x^2+y^2+x^4\sin^2 y}$$

7.3.3 多元复合函数的高阶偏导数

例 5 设 $w = f(x+y+z, xyz)$，f 具有二阶

连续偏导数，求 $\dfrac{\partial w}{\partial x}$ 和 $\dfrac{\partial^2 w}{\partial x \partial z}$.

解 令 $u = x+y+z$， $v = xyz$； $w = f(u,v)$

记 $f_1' = f_1 = \dfrac{\partial f}{\partial u}$

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = f_1' + yz f_2';$$

$$(= f_1 + yz f_2);$$

7.3.3 多元复合函数的高阶偏导数

例 5 设$w=f(x+y+z,xyz)$，f具有二阶

连续偏导数，求$\frac{\partial w}{\partial x}$和$\frac{\partial^2 w}{\partial x \partial z}$.

解 令 $u=x+y+z$, $v=xyz$; $w=f(u,v)$

注意 $f'_1=f_1=\frac{\partial f}{\partial u}$ 仍是关于u,v的函数

$f''_{12}=f_{12}=\frac{\partial^2 f}{\partial u \partial v}$, 同理有 f'_2, f''_{11}, f''_{22}.

f_2, f_{11}, f_{22}.

$u = x + y + z,$

$v = xyz;$

$$w = f(u,v) \qquad \frac{\partial w}{\partial x} = f_1' + yz f_2';$$

$$\frac{\partial^2 w}{\partial x \partial z} = \frac{\partial}{\partial z}(f_1' + yzf_2') = \frac{\partial f_1'}{\partial z} + yf_2' + yz\frac{\partial f_2'}{\partial z};$$

$$\frac{\partial f_1'}{\partial z} = \frac{\partial f_1'}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial f_1'}{\partial v}\cdot\frac{\partial v}{\partial z} = f_{11}'' + xyf_{12}'';$$

$$\frac{\partial f_2'}{\partial z} = \frac{\partial f_2'}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial f_2'}{\partial v}\cdot\frac{\partial v}{\partial z} = f_{21}'' + xyf_{22}'';$$

于是 $$\frac{\partial^2 w}{\partial x \partial z} = f_{11}'' + xyf_{12}'' + yf_2' + yz(f_{21}'' + xyf_{22}'')$$

$$= f_{11}'' + y(x+z)f_{12}'' + xy^2 zf_{22}'' + yf_2'.$$

例6　设$z = f(2x+y) + f(2x-y, y\sin x)$, 求 z_{xy}

解　$z_x = 2f'(2x+y) + f_1 \cdot 2 + f_2 \cdot y\cos x$

$z_{xy} = 2f''(2x+y) + 2(f_{11} \cdot (-1) + f_{12} \cdot \sin x) +$

$f_2 \cdot \cos x + y\cos x(f_{21} \cdot (-1) + f_{22} \cdot \sin x)$

$= 2f''(2x+y) - 2f_{11} + 2f_{12}\sin x + f_2\cos x$

$- yf_{21}\cos x + yf_{22}\sin x\cos x$

7.3.4 微分求导法 ——一阶微分形式不变性

设$z=f(u,v)$具有连续的偏导数，则有

$$dz=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv$$

又设$u=\varphi(x,y),v=\psi(x,y)$*也具有连续偏导时，*

则复合函数$z=f[\varphi(x,y),\psi(x,y)]$的全微分为：

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$$

$$=\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)dx+\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)dy$$

$$z = f(u,v),\quad u=\varphi(x,y), v=\psi(x,y)$$

$$= \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)$$

$$= \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv.$$

不论z是自变量u、v函数，或是中间变量u、v的函数，它的全微分形式是一样的，都是

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv$$

这个性质叫全微分形式的不变性.

利用这一性质，可求复合函数、隐函数的偏导数。

例 7 已知$e^{-xy}-2z+e^{z}=0$，求$\dfrac{\partial z}{\partial x}$和$\dfrac{\partial z}{\partial y}$.

解 $\because d(e^{-xy}-2z+e^{z})=0,$

$$\therefore e^{-xy}d(-xy)-2dz+e^{z}dz=0,$$

$$(e^{z}-2)dz=e^{-xy}(xdy+ydx)$$

$$dz=\frac{ye^{-xy}}{(e^{z}-2)}dx+\frac{xe^{-xy}}{(e^{z}-2)}dy$$

$$\frac{\partial z}{\partial x}=\frac{ye^{-xy}}{e^{z}-2},\qquad \frac{\partial z}{\partial y}=\frac{xe^{-xy}}{e^{z}-2}.$$

小结

本节主要讨论了多元复合函数的概念.

本节要求理解多元复合函数的概念；熟练掌握多元复合函数（特别是抽象函数）的一阶、二阶偏导数的计算.